# Birationally rigid varieties with a pencil of Fano double covers. I

**Aleksandr V. Pukhlikov**

Max-Planck-Institut für Mathematik
Vivatsgasse 7
53111 Bonn
GERMANY
e-mail: *pukh@mpim-bonn.mpg.de*

Steklov Institute of Mathematics
Gubkina 8
117966 Moscow
RUSSIA
e-mail: *pukh@mi.ras.ru*

Division of Pure Mathematics
Department of Mathematical Sciences
M&O Building, Peach Street
The University of Liverpool
Liverpool L69 7ZL
ENGLAND
e-mail: *pukh@liv.ac.uk*

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**Abstract**

We prove that a general Fano fibration $\pi: V \to \mathbb{P}^1$, the fiber of which is a double Fano hypersurface of index 1, is birationally superrigid provided it is sufficiently twisted over the base. In particular, on $V$ there are no other structures of a rationally connected fibration. The proof is based on the method of maximal singularities.

CONTENTS

# Introduction

In this paper we study birational geometry of higher-dimensional algebraic varieties with a pencil of Fano double covers. The main result of the paper, that is, the theorem on birational superrigidity of these varieties provided they are sufficiently twisted over the base, is formulated below in Sec. 0.3. The paper presents the outcome of the first stage of the research; the second part of this work, dealing with a relaxation of the twistedness condition, will be published in the subsequent paper.

## 0.1 Birationally rigid varieties

A rationally connected projective variety $V$ with at most $\mathbb{Q}$-factorial terminal singularities is said to be *birationally rigid*, if for any birational map

$$\chi\colon V - - \to V',$$

where $V'$ belongs to the same class of varieties, and any moving linear system $\Sigma'$ on $V'$ there exists a birational self-map $\chi^* \in \operatorname{Bir} V$, providing the following inequality

$$c(\Sigma) \leq c(\Sigma'), \tag{1}$$

where $\Sigma = (\chi \circ \chi^*)_* \Sigma'$ is the strict transform of the linear system $\Sigma'$ on $V$ with respect to the birational map

$$\chi \circ \chi^*\colon V \overset{\chi^*}{- - \to} V \overset{\chi}{- - \to} V',$$

and the symbol $c(\cdot)$ stands for the *threshold of the canonical adjunction* of the linear system $|\cdot|$,

$$c(\Lambda) = \sup\{\varepsilon \in \mathbb{Q}_+ \mid D + \varepsilon K \in A^1_+(\cdot)\},$$

$D \in \Lambda$ is an arbitrary divisor of the linear system $\Lambda$, $K$ stands for the canonical class of the variety, $A^1_+(\cdot) \subset A^1(\cdot) \otimes \mathbb{R}$ means the closed cone of effective cycles on the variety under consideration. The variety $V$ is said to be *birationally superrigid*, if the inequality (1) is always true for $\chi^* = \mathrm{id}_V$.

Let $\pi\colon V \to \mathbb{P}^1$ be a fibration into rationally connected varieties, where $V$ has $\mathbb{Q}$-factorial terminal singularities. (By the theorem of Graber-Harris-Starr [5], in this case the variety $V$ is itself automatically rationally connected.) The next question is of crucial importance for understanding birational geometry of $V$:

> are there other (that is, different from the original map $\pi\colon V \to \mathbb{P}^1$) structures of a rationally connected fibration on $V$?

Assume that $V$ is non-singular, $\operatorname{Pic} V = \mathbb{Z}K_V \oplus \pi^* \operatorname{Pic} \mathbb{P}^1$ and the following condition holds:

$$K_V \notin \mathrm{Int}\, A^1_+ V. \qquad (2)$$

Let $F_t = \pi^{-1}(t)$ be the fiber over a point $t \in \mathbb{P}^1$, $F \in \mathrm{Pic}\, V$ the class of a fiber. The condition (2) means that if $D \sim -nK_V + lF$ is an effective divisor on $V$, then $l \in \mathbb{Z}_+$. Conditions of this type for three-dimensional Mori fiber spaces are discussed in [2]. The following fact is well-known (see [17,20-22]).

**Proposition 0.1.** *In the assumptions above let $V$ be a birationally superrigid variety. Then:*

(i) *there is only one non-trivial structure of a fibration into rationally connected varieties on $V$, that is, the morphism $\pi$; in other words, if $\tau: W \to T$ is a fibration into rationally connected (or just uniruled) varieties and $\chi: V - - \to W$ is a birational map, then $\chi$ transform fibers into fibers, that is, the following diagram*

$$\begin{array}{ccccc} & V & \stackrel{\chi}{- - \to} & W & \\ \pi & \downarrow & & \downarrow & \tau \\ & \mathbb{P}^1 & \stackrel{\alpha}{- - \to} & T & \end{array}$$

*commutes for a certain map $\alpha: \mathbb{P}^1 \to T$.*

(i) *If $\tau: W \to T$ is another fibration of the same type, that is,* $\mathrm{Pic}\, W = \mathbb{Z}K_W \oplus \tau^* \mathrm{Pic}\, \mathbb{P}^1$, *and $\chi: V - - \to W$ is a (fiber-wise) birational map, then $\chi$ is an isomorphism of fibers of general position.*

Thus the property of being birationally rigid reduces birational geometry of the variety $V$ to biregular geometry of the fibration $V/\mathbb{P}^1$. That is why the word "rigidity" has been chosen: the variety $V$ does not admit any birational modifications inside the natural class of Fano fibrations with the relative Picard number one.

Nowadays quite a few classes of birationally (super)rigid Fano varieties are known (see, for instance, [3,14,18,19,23-25]). The known examples make it possible to conjecture that birational (super)rigidity is a typical property in dimension 3 and higher. Much less is known about Fano fibrations, their birational geometry is harder to investigate. A brief history of the theory of birational rigidity for Fano fibrations see below in Sec. 0.4. The aim of the present paper is to prove birational superrigidity of fibrations $V/\mathbb{P}^1$, the fibers of which are Fano double hypersurfaces of index 1 [19].

## 0.2 Varieties with a pencil of double covers

The symbol $\mathbb{P}$ stands for the projective space $\mathbb{P}^{M+1}$ over the field of complex numbers $\mathbb{C}$. Let

$$\mathcal{G} = \mathbb{P}(H^0(\mathbb{P}, \mathcal{O}_{\mathbb{P}}(m)))$$

be the space of all Fano hypersurfaces of degree $m$, $3 \leq m \leq M-1$, $\mathcal{W}$ the space of all hypersurfaces of degree $2l$ in $\mathbb{P}$, where $m + l = M + 1$. Let

$$\mathcal{F} = \{F \mid \sigma: F \stackrel{2:1}{\to} G\}$$

be the class of algebraic varieties realized as double covers of hypersurfaces $G \in \mathcal{G}$ branched over $W \cap G$, $W \in \mathcal{W}$. Set $\mathcal{F}_{sm} \subset \mathcal{F}$ to be the set of smooth double hypersurfaces corresponding to pairs $(G, W \cap G)$ of smooth varieties. Obviously, $F \in \mathcal{F}_{sm}$ is a smooth Fano variety of index 1 with the Picard group $\operatorname{Pic} F = \mathbb{Z}K_F$. Let $\mathcal{F}_{sm}^{reg} \subset \mathcal{F}_{sm}$ be the smooth subset consisting of varieties $F \in \mathcal{F}_{sm}$ satisfying the *regularity condition* of Sec. 1.3,1.4 below (which is identical to the regularity condition of Sec. 1.3 in [19]). Recall that in [19] the following fact was proved.

**Theorem A.** (i) *Any variety $F \in \mathcal{F}_{sm}^{reg}$ is birationally superrigid.*
(ii) *The set $\mathcal{F}_{sm}^{reg}$ is non-empty. Moreover, the following estimate holds:*

$$\operatorname{codim}_{\mathcal{F}_{sm}}(\mathcal{F}_{sm} \setminus \mathcal{F}_{sm}^{reg}) \geq 2.$$

Set $\mathcal{F}_{sing} = \mathcal{F} \setminus \mathcal{F}_{sm}$, $\operatorname{codim}_{\mathcal{F}} \mathcal{F}_{sing} = 1$. Let $\mathcal{F}_{sing}^{reg}$ be the open subset in $\mathcal{F}_{sing}$, consisting of all singular double hypersurfaces satisfying the regularity condition of Sec. 1.3,1.4 below. We note in Sec. 1.3,1.4 that the following inequality holds:

$$\operatorname{codim}_{\mathcal{F}}(\mathcal{F}_{sing} \setminus \mathcal{F}_{sing}^{reg}) \geq 2.$$

In the present paper we study Fano fibrations $V/\mathbb{P}^1$, each fiber $F_t = \pi^{-1}(t)$, $t \in \mathbb{P}^1$ of which is a variety from the family $\mathcal{F}$. Set

$$\mathcal{F}^{reg} = \mathcal{F}_{sm}^{reg} \cup \mathcal{F}_{sing}^{reg}.$$

By what was said above, $\operatorname{codim}_{\mathcal{F}}(\mathcal{F} \setminus \mathcal{F}^{reg}) \geq 2$. Since the fibration $V/\mathbb{P}^1$ can be looked at as a morphism $\mathbb{P}^1 \to \mathcal{F}$, that is, a curve in $\mathcal{F}$, for a general variety $V/\mathbb{P}^1$ we get:

$$F_t \in \mathcal{F}^{reg}$$

for all points $t \in \mathbb{P}^1$. If this is the case, we say that the fibration $V/\mathbb{P}^1$ is *regular*. A general construction of regular Fano fibrations $V/\mathbb{P}^1$ is described below in Sec. 1.2.

## 0.3 The main result

**Theorem 1.** *Assume that a regular fibration $V/\mathbb{P}^1$ satisfies the $K^2$-condition:*

$$K_V^2 \notin \operatorname{Int} A_+^2 V,$$

*where $A_+^2 V \subset A^2 V \otimes \mathbb{R}$ is the closed cone of effective cycles of codimension two. The the fibration $V/\mathbb{P}^1$ is birationally superrigid.*

The symbol $A^i\sharp$ stands, as usual, for the group of classes of codimension $i$ cycles on the variety $\sharp$ modulo numerical equivalence.

**Corollary 0.1.** (i) *For a fibration $V/\mathbb{P}^1$ of general position the following equality holds:*

$$\operatorname{Bir} V = \operatorname{Aut} V = \mathbb{Z}/2\mathbb{Z} = \{\mathrm{id}, \tau\},$$

*where $\tau \in \operatorname{Aut} V$ is the Galois involution of the double cover $V/Q$.*

(ii) *The variety $V$ is non-rational.*

Part (i) follows from Proposition 0.1 and Theorem A. Part (ii) is obvious.

We prove Theorem 1 in a few steps. Below in Sec. 1.1 we formulate a sufficient condition of birational superrigidity for an arbitrary Fano fibration $V/\mathbb{P}^1$ in terms of numerical geometry of fibers (Theorem 2). Essentially this fact was proved in [17,20], although the papers that we have just mentioned discussed Fano fibrations of a certain particular type. We will not repeat these arguments here, just making reference to [17,20].

Now to prove Theorem 1 we need to check that the fibers of the fibration $V/\mathbb{P}^1$, that is, the regular Fano hypersurfaces of index 1 (in the sense of the regularity conditions formulated below in Sec. 1.3,1.4) satisfy the conditions of Theorem 2. This verification makes our proof. It is carried out in Sections 2 and 3. Birational geometry of varieties with a pencil of Fano double covers that do not satisfy the $K^2$-condition will be studied in the next paper, the second part of the present research.

**Remark.** Superrigidity of Fano fibrations $V/\mathbb{P}^1$, the fibers of which are double spaces ($m = 1$) and double quadrics ($m = 2$) of index 1, is proved in [17] and [21,22], respectively, and for this reason these varieties are not considered in this paper.

## 0.4 Historical remarks

Investigating structures of a fibration into rationally connected (uniruled) varieties is a very old subject. The classical proof of the Noether theorem on the Cremona group of the plane, presented by Yu.I.Manin in [1], can be looked at in this way: step by step a certain pencil of rational curves on $\mathbb{P}^2$ is modified, thus one goes over from one structure of a $\mathbb{P}^1$-fibration on the plane to another. Birational geometry of varieties of dimension higher than two, on which there are a lot of various structures of a rationally connected fibration, is very hard to study. Apart from a few exceptional types, these varieties are still out of reach for the modern technique. However, on rationally connected varieties which are in a certain sense general there is only one structure of a rationally connected fibration with the minimality condition that it is equivalent to a Fano fibration with the relative Picard number one. This is the very phenomenon of birational rigidity.

In the modern birational geometry birationally rigid varieties first come to the light in the papers of Yu.I.Manin in the form of del Pezzo surfaces over non-closed fields [15,16]. The first theorems on birational rigidity of non-trivial rationally connected fibrations were proved by V.A.Iskovskikh as theorems on uniqueness of a pencil of rational curves for some surfaces over non-closed fields. These theorems continue the above-mentioned work of Yu.I.Manin. The study of the absolute and relative cases in dimension two over a non-closed field prepared the basis for working in higher dimensions.

In the classical paper of V.A.Iskovskikh and Yu.I.Manin [14] the *test class* technique was developed that made it possible to prove (in the modern terminology) birational superrigidity of the smooth three-dimensional quartic $V_4 \subset \mathbb{P}^4$ (actually, in [14] birational superrigidity of the double space branched over a sextic was proved

as well and also the crucial step was made in the proof of birational rigidity of the double quadric of index one [11]). After that attempts were made to use this technique to obtain similar results in the relative case for fibrations over a non-trivial base. For one class of varieties the study was successful: V.G.Sarkisov's theorem proves that the given structure of a conic bundle is unique provided the discriminant divisor is sufficiently big [26,27]. The proof of Sarkisov's theorem is based on the following two technical principles:

(1) the test class technique of V.A.Iskovskikh and Yu.I.Manin,

(2) the fiber-wise modifications.

The possibility of making fiber-wise modifications of conic bundles remaining at the same time in the class of smooth varieties is an exclusive property of these varieties. In a sense this special feature comes from the fact that the group of automorphisms of the fiber $\mathrm{Aut}\,\mathbb{P}^1$ is very big: say, for a typical Fano variety of dimension higher than two this group is finite. Thus there is no hope to use similar arguments for higher dimensional Fano fibrations.

After Sarkisov's papers [26,27] had been published, there remained only one class of rationally connected three-folds, birational geometry of which was a terra incognita, that is, the class of fibrations into del Pezzo surfaces over $\mathbb{P}^1$. The attempts to use fiber-wise modifications similar to Sarkisov's theorem proved unsuccessful, since immediately converted the variety under consideration into a singular one and, moreover, the acquired singularities were out of control. However, the above-mentioned test class technique also refused to work. Since mid-80s and up to mid-90s attempts were made to construct at least some examples of three-dimensional del Pezzo fibrations similar to Sarkisov's rigid conic bundles, but without any success. This activity was summed up in [12]: the only outcome of the almost decade-long work and an immense amount of completed computations were some conjectures — and no essential progress in their proof. One can see from [12] that there was no understanding why the test class technique that works so impeccably in the absolute case (the three-dimensional quartic [14]) does not allow a single step forward in the case of del Pezzo fibrations: the test class simply refused to be constructed.

The situation changed radically when the paper [17] appeared. It became immediately clear why the test class technique refused to generalize to the relative case: as it turned out, the desired class just did not exist. For the three-dimensional quartic the test class technique is equivalent to the technique of counting multiplicities introduced in [17,18,23]. However, the technique of counting multiplicities is much more flexible, since it describes properties of a certain effective cycle of codimension two (the self-intersection of the moving linear system defining the birational map under consideration), whereas the test class gives just a number, the intersection number of this cycle with the test class. In the relative case, when the base of the fibration is non-trivial, any effective cycle can be decomposed into the vertical and horizontal components. Informally speaking, each of them requires its own test class.

The methods developed in the paper [17] were later used for proving birational rigidity of big classes of higher-dimensional Fano fibrations [20-22]. In these papers (and in the present paper as well) birational rigidity is derived from the $K^2$-

condition. However when the $K^2$-condition is somewhat weakened the methods of these papers still work well and make it possible to give a complete description of birational geometry of the variety under consideration. See [29,30] and the series of papers [6-8], where birational rigidity is proved for a few classes of del Pezzo fibrations over $\mathbb{P}^1$. These classes were not considered in [17] because they do not satisfy the $K^2$-condition. However, when the deviation from the $K^2$-condition grows too strong, the methods fail to work.

Note also that in spite of the progress in the general theory of factorization of birational maps between three-fold Mori fiber spaces (the Sarkisov program [4,28]), all attempts either to improve Sarkisov's results or to prove the rationality criterion for conic bundles have been unsuccessful up to this day [13]. See the recent paper [2] on this point. We will discuss it in the next papers.

### 0.5 Acknowledgements

One part of the present research (corresponding to the second section of this paper) was carried out by the author during his stay at the University of Bayreuth in 2001 as a Humboldt Research Fellow. The crucial step (investigation of singularities coming from a double point on the branch divisor of a fiber, making the contents of the third section of the paper) was made during my work at Max-Planck-Institut für Mathematik in Bonn in 2003. The author is very grateful to Alexander von Humboldt Stiftung, Mathematisches Institut der Universität Bayreuth (in the first place, to Prof. Th.Peternell) and Max-Planck-Institut für Mathematik in Bonn for hospitality, the excellent conditions of work and general support.

## 1 The method of maximal singularities and the regularity conditions

### 1.1 A criterion of birational rigidity

Let $\pi: V \to \mathbb{P}^1$ be a smooth standard Fano fibration, that is, $V$ be a smooth variety with

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F,$$

where $F$ is the class of a fiber. Define the *degree* of a horizontal subvariety $Y \subset V$, $\pi(Y) = \mathbb{P}^1$, by the formula

$$\deg Y = (Y \cdot F \cdot (-K_V)^{\dim Y - 1}),$$

and the degree of a vertical subvariety $Y \subset \pi^{-1}(t)$ by the formula

$$\deg Y = (Y \cdot (-K_V)^{\dim Y}).$$

By this definition the degree of the variety $V$ itself coincides with the degree of a fiber, $\deg V = \deg F$.

Smooth Fano fibrations, the fibers of which are complete intersections in weighted projective spaces, satisfy also the following property: their fibers have at most isolated singularities.

We say that the Fano fibration $V/\mathbb{P}^1$ satisfies

*condition* (v), if for any irreducible vertical subvariety $Y$ of codimension 2, $Y \subset \pi^{-1}(t) = F_t$, and any smooth point $o \in F_t$ the following estimate holds:

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{2}{\deg V};$$

*condition* (vs), if for any vertical subvariety $Y \subset F_t$ of codimension 2 (with respect to $V$, that is, a prime divisor on $F_t$), any singular point $o \in F_t$ and any infinitely near point $x \in \widetilde{F}_t$, where $\varphi\colon \widetilde{F}_t \to F_t$ is the blow up of the point $o$, $\varphi(x) = o$, $\widetilde{Y} \subset \widetilde{F}_t$ the strict transform of the subvariety $Y$ on $\widetilde{F}_t$, the following estimates hold:

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{4}{\deg V}, \quad \frac{\mathrm{mult}_x \widetilde{Y}}{\deg Y} \leq \frac{2}{\deg V};$$

*condition* (h), if for any horizontal subvariety $Y$ of codimension 2 and any point $o \in Y$ the following estimate holds

$$\frac{\mathrm{mult}_o}{\deg} Y \leq \frac{4}{\deg V}.$$

Assume that $\dim V \geq 4$ and the variety $V$ satisfies the condition

$$A^2 V = \mathbb{Z} K_V^2 \oplus \mathbb{Z} H_F,$$

where $H_F = (-K_V \cdot F)$ and a fiber $F = F_t \subset V$ of general position satisfies the condition $A^2 F = \mathbb{Z}(H_F \cdot H_F)_F$. Set $A^2_{\mathbb{R}} V = A^2 V \otimes \mathbb{R} \cong \mathbb{R}^2$ and define the cone of effective cycles $A^2_+ V \subset A^2_{\mathbb{R}} V$ as the closure (in the real topology) of the set

$$\{\lambda \Delta \mid \lambda \in \mathbb{R}_+, \Delta \text{ is the class of an effective cycle}\}.$$

**Definition 1.1** We say that the Fano fibration $V/\mathbb{P}^1$ satisfies the $K^2$-*condition*, if

$$K_V^2 \notin \mathrm{Int}\, A^2_+ V.$$

**Remark.** It is easy to see that the $K^2$-condition is equivalent to the following claim: for any $a \geq 1$ and $b \geq 1$ the class

$$\Delta(a, b) = a K_V^2 - b H_F$$

is not effective. Indeed, $H_F \in A^2_+V$, so that $K_V^2 \in \operatorname{Int} A^2_+V$ if and only if $\Delta(N,1) \in A^2_+V$ for some $N \geq 1$. This implies immediately that both conditions are equivalent.

**Theorem 2.** *Assume that the smooth standard Fano fibration $V/\mathbb{P}^1$ satisfies the $K^2$-condition and the conditions* (v), (vs) *and* (h). *Then $V/\mathbb{P}^1$ is birationally superrigid.*

For the **proof** see [17,20].

## 1.2 An explicit construction of the fibration $V/\mathbb{P}^1$

Let us describe an explicit construction of regular fibrations $V/\mathbb{P}^1$. For each fiber $F \in \mathcal{F}^{reg}$ (singular or smooth) the anticanonical linear system $|-K_F|$ determines precisely the double cover $\sigma_F\colon F \to G \subset \mathbb{P}$. For this reason, $\pi_*\mathcal{O}(-K_V)$ is a locally free sheaf of rank $M+2$ on $\mathbb{P}^1$. It gives a locally trivial $\mathbb{P}$-fibration over $\mathbb{P}^1$. The variety $V$ is realized as a double cover of a smooth divisor $Q$ on $\mathbb{P}(\pi_*\mathcal{O}(-K_V))$. Namely, let

$$\mathcal{E} = \bigoplus_{i=0}^{M+1} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

be a locally free sheaf, normalized by the condition that

$$a_0 = 0 \leq a_1 \leq \ldots \leq a_i \leq a_{i+1} \leq \ldots \leq a_{M+1}.$$

In particular, $\mathcal{E}$ is generated by global sections. Set $X = \mathbb{P}(\mathcal{E})$ to be its **Proj** in the sense of Grothendieck, $\pi_X\colon X \to \mathbb{P}^1$ the natural projection, $\mathcal{L}_X$ the tautological sheaf, $Q \subset X$ a smooth divisor on $X$, corresponding to a section

$$s_Q \in H^0\left(X, \mathcal{L}_X^{\otimes m} \otimes \pi_X^*\mathcal{O}_{\mathbb{P}^1}(a_Q)\right),$$

$a_Q \in \mathbb{Z}_+$. The symbol $\pi_Q\colon Q \to \mathbb{P}^1$ stands for the projection $\pi_X|_Q$. Obviously, $Q/\mathbb{P}^1$ is a smooth fibration into Fano hypersurfaces of degree $m$ in $\mathbb{P}$. Let $W \subset X$ be an irreducible hypersurface, corresponding to a section

$$s_W \in H^0\left(X, \mathcal{L}_X^{\otimes 2l} \otimes \pi_X^*\mathcal{O}_{\mathbb{P}^1}(2a_W)\right),$$

$a_W \in \mathbb{Z}_+$, whereas $W_Q = W \cap Q$ is a smooth divisor on $Q$. We denote the fiber $\pi_Q^{-1}(t)$ over a point $t \in \mathbb{P}^1$ by the symbol $G_t$ (or just $G$, when it is clear which point is meant or when it is inessential). Finally, set

$$\sigma\colon V \to Q$$

to be the double cover, branched over $W_Q$. The natural projection onto $\mathbb{P}^1$ will be denoted by $\pi$, the fiber $\pi^{-1}(t)$ by the symbol $F_t$ (or just $F$). It is easy to see that

$$\operatorname{Pic} V = \mathbb{Z}K_V \oplus \mathbb{Z}F$$

and up to twisting by an invertible sheaf $\mathcal{O}_{\mathbb{P}^1}(k)$, $k \in \mathbb{Z}$, the sheaves $\mathcal{E}$ and $\pi_*\mathcal{O}(-K_V)$ on $\mathbb{P}^1$ coincide.

More precisely, let $L_X \in \operatorname{Pic} X$ be the class of the tautological sheaf $\mathcal{L}_X$, $L_Q = L_X|_Q$ its restriction to $Q$, so that

$$\operatorname{Pic} Q = \mathbb{Z}L_Q \oplus \mathbb{Z}G.$$

Set $L_V = \sigma^* L_Q$. It is easy to see that

$$K_V = -L_V + (a_1 + \ldots + a_M - 2 + a_Q + a_M)F.$$

By the Lefschetz theorem

$$A^2 V = \mathbb{Z}K_V^2 \oplus \mathbb{Z}H_F,$$

where $H_F = (-K_V \cdot F)$ is the class of a hyperplane section. The symbol $H_F$ is used in the present paper in two different meanings: as a class of codimension two on $V$ and as the hyperplane section of the fiber, that is, an element of $A^1 F$. Every time it is clear which of the two concepts is meant.

It is easy to compute that

$$(K_V^2 \cdot L^{M-1}) = 2m(4 - a_1 - \ldots - a_{M+1} - a_Q - a_W) + 2a_Q.$$

Since $(H_F \cdot L^{M-1}) = 2m$ and the linear system $|L_V|$ is free, the inequality $(K_V^2 \cdot L^{M-1}) \leq 0$ implies, that $K_V^2 \notin \operatorname{Int} A_+^2 V$, where $A_+^2 V \subset A^2 V \otimes \mathbb{R}$ is the closed cone of effective cycles of codimension two.

## 1.3 The regularity conditions outside the branch divisor

Let $\sigma: F \to G \subset \mathbb{P}$ be a Fano double hypersurface of index 1, $F \in \mathcal{F}$. The variety $F$ is realized as a complete intersection of codimension two in the weighted projective space

$$\mathbb{P}(\underbrace{1, 1, \ldots, 1,}_{M+2} l),$$

see [19]: $F$ is of type $m \cdot 2l$ and given by the pair of equations

$$\tilde{f}(x_0, \ldots, x_{M+1}) = 0, \quad u^2 = \tilde{g}(x_0, \ldots, x_{M+1}),$$

where $x_*$ are the coordinates of weight 1, $u$ is the coordinate of weight $l$, $\tilde{f}$ is the equation of the hypersurface $G \subset \mathbb{P} = \mathbb{P}(1, \ldots, 1)$, $\tilde{g}$ is the equation of the hypersurface $W \cap \mathbb{P}$.

Let $o \in F$ be an arbitrary point. First of all, we draw the reader's attention to the following obvious fact:

$$o \neq (\underbrace{0, 0, \ldots, 0,}_{M+2} 1).$$

Thus we may assume that the point $o$ lies in one of the standard affine charts $\mathbb{A}^{M+2}$ with the coordinates

$$z_i = x_i / x_0, \quad i = 1, \ldots, M+1, \quad y = u / x_0^l$$

and its $z_*$-coordinates are $(0,\dots,0)$. With respect to the coordinate system $(z_*,y)$ the affine part of the variety $F$ is given by the pair of equations

$$f=q_1+\dots+q_M=0,\quad y^2=g=w_0+\dots+w_{2l},$$

where $q_i$ and $w_j$ are homogeneous polynomials in $z_*$ of degrees $i$ and $j$, respectively. Set $p=\sigma(o)\in G$. The point $p$ lies on the branch divisor $W$ if and only if $w_0=0$. If $p\notin W$, then we normalize the second equation and assume that $w_0=1$.

Let us formulate first the regularity conditions outside the branch divisor. In this case the fiber $F$ is given with respect to the affine coordinate system $(z_*,y)$ with the origin of the $z_*$-system at $p=\sigma(o)$ by the equations

$$\left\{\begin{array}{l} f=q_a+\dots+q_m=0,\\ y^2=g=1+w_1+\dots+w_{2l}, \end{array}\right.$$

where $a\geq 1$.

Set

$$\sqrt{g}=(1+w_1+\dots+w_{2l})^{1/2}=1+\sum_{i=1}^{\infty}\gamma_i(w_1+\dots+w_{2l})^i=$$

$$=1+\sum_{i=1}^{\infty}\Phi_i(w_1,\dots,w_{2l}),$$

where $\Phi_i(w_1(z_*),\dots,w_{2l}(z_*))$ are homogeneous in $z_*$ of degree $i\geq 1$,

$$\gamma_i=(-1)^{i-1}\frac{(2i-3)!!}{2^i i!}=(-1)^{i-1}\frac{(2i-3)!}{2^{2i-2}i!(i-2)!}$$

is the standard $i$-th coefficient of the Taylor expansion of the function $(1+s)^{1/2}$ at the point $s=0$. Obviously,

$$\Phi_i(w_*)=w_i+A_i(w_1,\dots,w_{i-1})$$

for $i\leq 2l$. For $i\geq 1$ set

$$[\sqrt{g}]_i=1+\sum_{j=1}^{i}\Phi_i(w_*),\quad g^{(i)}=g-[\sqrt{g}]_i^2.$$

It is easy to see that the first non-zero component of the polynomial $g^{(i)}$ is of degree $i+1$. More precisely, this component is equal to

$$g_{i+1}=2\Phi_{i+1}(w_1(z_*),\dots,w_{i+1}(z_*)).$$

**The regularity condition at a smooth point** $p\in G$ (R1.1):

The sequence

$$q_1, \dots, q_m, g_{l+1}, \dots, g_{2l-1}$$

is regular in $\mathcal{O}_{p,\mathbb{P}}$. Here $a = 1$.

**The regularity condition at a double point** $p \in G$ (R1.2):

If $2l \geq m+1$, then the system of $M-1$ homogeneous polynomials

$$q_2, \dots, q_m, g_{l+1}, \dots, g_{2l-1},$$

whereas if $2l \leq m$, then the system of homogeneous polynomials

$$q_2, \dots, q_{m-1}, g_{l+1}, \dots, g_{2l}$$

defines a curve in $\mathbb{P}^M = \mathbb{P}(T_p\mathbb{P})$, neither component of which is contained in a hyperplane.

Furthermore, the system of $M$ homogeneous equations

$$q_2 = \dots = q_{m-1} = g_{l+1} = \dots = g_{2l} = 0 \tag{3}$$

defines a non-zero subscheme $Z_*$ in $\mathbb{P}^M$, such that for any hyperplane $P \subset \mathbb{P}^M$

$$\deg(P \cap Z_*) < \lambda_{m,l} = \frac{m!(2l-1)!}{6(l-1)!}$$

for $m \geq 4$ and

$$\deg(P \cap Z_*) < \lambda_{3,l} = 12\frac{(2l-1)!}{(l+1)!}(l-2)$$

for $m = 3$. If the scheme $Z_*$ is reduced, then this condition means simply that any set of $\lambda_{m,l}$ points is not contained in a hyperplane.

**Remark.** Since $w_0 = y(0) = 1$, in a neighborhood of the singular point $o \in F$ the equations

$$y - [\sqrt{g}]_i = 0 \quad \text{and} \quad \sigma^* g^{(i)} = 0$$

define the same divisor. Consider the system of equations (3) on the fiber $F$ (and not on the projectivized tangent space $\mathbb{P}(T_p\mathbb{P})$). The system defines an effective 1-cycle $C_*$ on $F$. By construction, its degree is equal to

$$\deg C_* = 2m!\frac{(2l-1)!}{(l-1)!},$$

whereas its multiplicity at the point $o \in F$ satisfies the estimate

$$\operatorname{mult}_o C_* \geq m!\frac{(2l)!}{l!} = \deg C_*, \tag{4}$$

so that what we actually have in (4) is an equality and $C_*$ is an algebraic sum of lines on $F$, that is, curves of the form $L \ni o$, the image $\sigma(L) \subset \mathbb{P}$ of which is a line, and moreover the morphism $\sigma: L \to \sigma(L)$ is an isomorphism. Considering the zero-dimensional scheme $Z_*$ as an effective zero-dimensional cycle, we get by construction:

$$Z_* = \mathbb{P}(T_o C_*).$$

In particular, for any hyperplane $P \subset \mathbb{P}$ the one-dimensional part of the scheme

$$\{q_2 = \ldots = q_m = g_{l+1} = \ldots = g_{2l} = 0\} \cap \sigma^{-1}(P)$$

is of degree not higher than $\lambda_{m,l} - 1$. In other words, if all components of the cycle $C_*$ are of multiplicity 1, then no more than $\lambda_{m,l} - 1$ of these lines are contained in $\sigma^{-1}(P)$.

## 1.4 The regularity conditions on the branch divisor

In this case the variety $F$ is given with respect to the affine coordinate system $(z_*, y)$ by the system of equations

$$\left\{ \begin{array}{l} f = q_1 + \ldots + q_m = 0, \\ y^2 = g = w_1 + \ldots + w_{2l}. \end{array} \right.$$

**The regularity condition at a smooth point** $o \in F$ (R2.1):

the sequence of homogeneous polynomials

$$q_1, \ldots, q_m$$

is regular in $\mathcal{O}_{p,\mathbb{P}}$ and the quadratic form $q_2$ does not vanish identically on the plane $\{q_1 = w_1 = 0\}$.

Note that since the point $o \in F$ is smooth, this plane is of codimension exactly two, that is, the linear forms $q_1$ and $w_1$ are linearly independent: the plane $\{q_1 = w_1 = 0\}$ is the tangent plane to the branch divisor $W \cap G$ of the morphism $\sigma_F$.

**The regularity condition at a double point** $o \in F$ (R2.2):

In this case we have the double cover $\sigma_F: F \to G$, branched over the divisor $W_G = W \cap G$. The first regularity condition is smoothness of the hypersurface $G$ at the point $p = \sigma(o)$, that is, $q_1 \neq 0$. Furthermore, the divisor $W_G$ should have at the point $p$ a non-degenerate quadratic singularity:

$$w_1 = \lambda q_1,$$

$\lambda \in \mathbb{C}$. For convenience of notations assume that $q_1 = z_{M+1}$. The quadratic polynomial

$$\bar{w}_2 = w_2|_{\{z_{M+1}=0\}}$$

is of the maximal rank. Let $E_G \cong \mathbb{P}^{M-1}$ be the exceptional divisor of the blow up $\varphi_G\colon \widetilde{G} \to G$ of the point $p$. Take $z_1, \ldots, z_M$ for homogeneous coordinates on $E_G$ and set

$$W_E = \{\bar{w}_2 = 0\}.$$

It is a non-singular quadratic hypersurface in $E_G$. Denote by the symbol $\bar{q}_i$ the restriction of the homogeneous polynomial $q_i$ onto the hyperplane $q_{M+1} = 0$. Now the remaining part of the condition (R2.2) looks as follows:

the system of homogeneous equations

$$\bar{q}_2 = \ldots = \bar{q}_m = 0$$

defines in $E_G \cong \mathbb{P}^{M-1}_{(z_1:\ldots:z_M)}$ an irreducible subvariety $Z_{2\cdot\ldots\cdot m}$, which is an irreducible reduced complete intersection of codimension $(m-1)$. The quadric

$$\bar{q}_2 = 0$$

is smooth and distinct from $W_E$.

**Definition 1.2.** A Fano double hypersurface $F \in \mathcal{F}$ is *regular*, if each smooth point on it is regular in the sense of the corresponding condition (R1.1) or (R2.1) and each of its singular points is regular in the sense of the corresponding condition (R1.2) or (R2.2). Notation: $F \in \mathcal{F}^{reg}$.

The conditions (R1.1) and (R2.1) coincide with the regularity conditions of the paper [19] (Definitions 1 and 2 in Sec. 1.3). In [19, Sec. 4.3] it was shown that non-regular smooth double spaces form a closed subset of codimension at least two in the set of all smooth double hypersurfaces $\mathcal{F}_{sm}$. Moreover, it follows from the computations of Sec. 4.3 in [19] that the set of Fano double hypersurfaces $F$ with at least one *smooth* non-regular point $o \in F$ is of codimension at least two in $\mathcal{F}$. Thus a general singular double hypersurface $F \in \mathcal{F}_{sing}$ has exactly one singular point whereas all its smooth points are regular. The singular point $o \in F$ is a non-degenerate double point. If $p = \sigma(o) \notin W_G$, then the fact that the condition (R1.2) is open implies that in a neighborhood of $F \in \mathcal{F}$ the following estimate holds

$$\operatorname{codim}_{\mathcal{F}_{sing}}(\mathcal{F}_{sing} \setminus \mathcal{F}^{reg}_{sing}) \geq 1 \tag{5}$$

and thus

$$\operatorname{codim}_{\mathcal{F}}(\mathcal{F}_{sing} \setminus \mathcal{F}^{reg}_{sing}) \geq 2. \tag{6}$$

If $p = \sigma(o) \in W_G$, then in a similar way the fact that the condition (R2.2) is open implies the estimate (5) in a neighborhood of $F \in \mathcal{F}$. Thus the estimates (5) and (6) are global.

## 1.5 Start of the proof of Theorem 1

Let us check that the regular fibration $V/\mathbb{P}^1$ satisfies the conditions (v) and (h). Assume that the opposite inequality holds:

$$\frac{\mathrm{mult}_o}{\deg} Y > \frac{2}{\deg V},$$

where $o \in F = F_t$ is a smooth point, $Y \subset F$ is a prime divisor. Let

$$T = \sigma^{-1}(T_p G \cap G)$$

be the tangent divisor, $p = \sigma(o)$. By the regularity conditions, $\mathrm{mult}_o T = 2$. Since $T \subset F$ is a hyperplane section, we get $\deg T = \deg V$, so that

$$\frac{\mathrm{mult}_o}{\deg} T = \frac{2}{\deg V}$$

and thus $Y \neq T$. Both subvarieties $Y$, $T$ are irreducible, so that the intersection $Y \cap T$ is of codimension two with respect to $F$ and the effective cycle $Z = (Y \circ T)$ of the scheme-theoretic intersection of $Y$ and $T$ is well defined. Obviously, the cycle $Z$ satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Z > \frac{4}{\deg V}. \tag{7}$$

However it was proved in [19], Sec. 3, that for a regular point $o \in F$ it is impossible. This proves the condition (v).

Let us prove that the condition (h) holds. To begin with, let us consider first the smooth case, where $o \in F$ is a smooth point. Assume that an irreducible horizontal subvariety $Y \subset V$ of codimension two satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Y > \frac{4}{\deg V}.$$

Since $\pi(Y) = \mathbb{P}^1$, we get $Y \neq F$, so that $Z = (Y \circ F)$ is an effective cycle of codimension two on the fiber $F$, satisfying the inequality (7). As it was pointed out above, this is impossible. The condition (h) is proved in the smooth case.

Now let $o \in F$ be a double point. Arguing in the same way as in the smooth case, let us construct the effective cycle $Z = (Y \circ F)$ of codimension two on the fiber $F$. Since $\frac{\mathrm{mult}_o}{\deg} F = 2$, the cycle $Z$ satisfies the inequality

$$\frac{\mathrm{mult}_o}{\deg} Z > \frac{8}{\deg V}.$$

Let us show that this is impossible. Without loss of generality assume that $Z \subset F$ is an irreducible subvariety of codimension two. Its image on $G$ satisfies the estimate

$$\frac{\mathrm{mult}_p}{\deg} \sigma(Z) > \frac{4}{\deg G}.$$

Now if $p \in G$ is a smooth point, then the arguments of the paper [20] (they work without any modifications for an arbitrary degree $\deg G \leq \dim G + 1$) show that this is impossible. If $p \in G$ is a double point, then by the condition (R1.2) the homogeneous polynomials $q_2, \dots, q_m$ make a regular sequence, so that the standard arguments of [20] give a contradiction once again (see Sec. 3.1 in [20]).

This completes the proof of the condition (h).

# 2 Singularity of a fiber outside the branch divisor

## 2.1 Hypertangent divisors and linear systems

Let $\varphi = \varphi_{F,o}\colon \widetilde{F} \to F$ be the blow up of the fiber at an arbitrary point $o$, $\varphi_G = \varphi_{G,p}\colon \widetilde{G} \to G$ the blow up of the fiber $G$ at the point $p = \sigma(o)$, $E = E_F \subset \widetilde{F}$ and $E_G \subset \widetilde{G}$ the exceptional divisors.

**Definition 2.1.** The linear system

$$\varphi_*(|kH_F - (k+1)E|)$$

of divisors on $F$ (respectively, the linear system

$$(\varphi_G)_*(|kH_G - (k+1)E_G|)$$

of divisors on $G$) is called the *$k$-th hypertangent linear system* and denoted by the symbol $\Lambda_k = \Lambda_k^F$ (respectively, $\Lambda_k^G$).

One can say that $\Lambda_k$ is the largest linear subsystem of the system $|kH_F|$, the strict transform of which satisfies the property

$$\widetilde{\Lambda}_k \subset |kH_F - (k+1)E|,$$

and similarly for $G$. In the general case one cannot assert that

$$\sigma^* \Lambda_k^G \subset \Lambda_k, \tag{8}$$

since if $p \in W_G$ is a smooth point of the branch divisor, then the double cover $\sigma\colon F \to G$ *does not* extend to a double cover $\widetilde{F} \to \widetilde{G}$ (there is a rational map of degree two between these varieties; this rational map has a fairly simple structure, however it is not a finite morphism). But if $p \notin W_G$ or $p \in W_G$ is a double point of the branch divisor, then the inclusion (8) holds.

The symbol

$$\Lambda_k^E$$

stands for the corresponding linear system on the exceptional divisor:

$$\Lambda_k^E = \widetilde{\Lambda}_k|_E \quad \text{or} \quad \Lambda_k^E = \widetilde{\Lambda}_k^G|_{E_G},$$

depending on the context. It is easy to see that

$$(\widetilde{\operatorname{Bs} \Lambda_k} \circ E) = \operatorname{Bs} \Lambda_k^E$$

in the scheme-theoretic sense, in particular, the corresponding effective algebraic cycles are equal, that is, the equality respects multiplicities.

Abusing out notations, we sometimes use the notion of a hypertangent system for a certain special subsystem of the hypertangent system, which permits an explicit description. In practice it is these special subsystems that we use. Let $p \in G$ be a point, $z_1, \dots, z_{M+1}$ a system of linear coordinates with the origin at $p$, and assume that the hypersurface $G$ is given by the equation

$$f = q_a + q_{a+1} + \ldots + q_m = 0,$$

$a = 1$ or 2. Then

$$\Lambda_k^G \supset \left| \sum_{i=a}^{k} s_{k-i} f_i \right|, \qquad (9)$$

where

$$f_i = q_a + \ldots + q_i,$$

$k \geq a$ and $s_j$ means an arbitrary homogeneous polynomial of degree $j$ in the variables $z_*$. The inclusion (9) is obvious, since

$$f_i|_G = (-q_{i+1} - \ldots - q_m)|_G.$$

Now assume that $p \notin W_G$. Let us construct the hypertangent system $\Lambda_k$. Obviously, $\Lambda_k \supset \sigma^* \Lambda_k^G$, but in fact the system $\Lambda_k$ is much larger. Following [19,22,25], let us describe the construction of hypertangent divisors, associated with the double cover $\sigma$. Since $p \notin W_G$, we may assume that the hypersurface

$$W_t = W \cap \mathbb{P}_t \subset \mathbb{P}$$

is given by the equation

$$g(z_*) = 1 + w_1 + \ldots + w_{2l} = 0,$$

$w_i(z_*)$ are homogeneous of degree $i$. Setting formally

$$\sqrt{g} = 1 + \sum_{i=1}^{\infty} \Phi_i(w_1, \dots, w_{2l}), \qquad (10)$$

where $\Phi_i(w_1(z_*), \dots, w_{2l}(z_*))$ are homogeneous polynomials of degree $i$ in $z_*$, write for $j \geq 1$

$$[\sqrt{g}]_j = 1 + \sum_{i=1}^{j} \Phi_i(w_*(z_*)).$$

Now we get

$$\Lambda_k \supset \left| \sum_{i=a}^{k} s_{k-i} f_i + \sum_{i=l}^{\min\{k,2l-1\}} s^*_{k-i}(y - [\sqrt{g}]_i) \right|, \qquad (11)$$

where $s^*_{k-i}$ are homogeneous polynomials in $z_*$ of degree $k-i$; if $k \leq l-1$, then the right-hand side is assumed to be equal to zero. The inclusion (11) follows from (9) and the following fact.

**Lemma 2.1.** *In the local coordinates $z_*$ we get*

$$(y - [\sqrt{g}]_i)|_F = 2\Phi_{i+1}(w_*(z_*))|_F + \ldots,$$

*where the dots stand for a formal series, the components of which are homogeneous polynomials of degree $i+2$ and higher in the variables $z_*$.*

**Proof:** it is obvious, since $(y^2 - g)|_F \equiv 0$, $g(p) = 1$ and the formal decomposition (10) holds.

Note that

$$\Phi_i(w_*) = \frac{1}{2} w_i + A_i(w_1, \ldots, w_{i-1}).$$

Now let us consider the case when $p = \sigma(o) \in W_G$. If the branch divisor is non-singular at the point $p$, then the local equation of the hypersurface $W_t$ is of the form

$$g(z_*) = w_1 + \ldots + w_{2l} = 0,$$

where the linear forms $q_1$, $w_1$ are linearly independent. Since the inverse image of the divisor

$$\{w_1|_G = 0\}$$

on $F$ is obviously singular, we obtain:

$$\Lambda_k \supset \left| \sum_{i=1}^{k} s_{k-i} f_i + s_{k-1} w_1 \right|. \qquad (12)$$

However, if $p = \sigma(o)$ is a singularity of the divisor $W_G$, then our methods of constructing hypertangent linear systems give at most the inclusion $\Lambda_k \supset \sigma^* \Lambda_k^G$, where $\Lambda_k^G$ is given by the formula (9).

The regularity conditions make it possible to get a lower bound for the codimension of the base set of hypertangent systems. In the formulae below it is assumed that the segment $[a, b] \subset \mathbb{R}$ is an empty set when $b < a$. For an arbitrary point $o \in F$ set

$$\mathcal{M} = [a, m-1] \cap \mathbb{Z}_+ = \{a, \ldots, m-1\},$$

where $a = \operatorname{mult}_p G \in \{1, 2\}$, $p = \sigma(o)$, and

$$\mathcal{L} = [l, 2l + a - 3] \cap \mathbb{Z}_+ = \{l, \ldots, 2l + a - 3\}.$$

Thus the sets $\mathcal{M}$, $\mathcal{L}$ depend on the type of the point $o$. At each stage of the proof the point $o$ is assumed to be fixed and the symbols $\mathcal{M}$, $\mathcal{L}$ mean the sets corresponding to this point.

For $e = \max\{m-1, 2l-1\}$ we denote the hypertangent linear system $\Lambda_e$ by the symbol $\Lambda_\infty$.

**Proposition 2.1.** *The following estimates hold:*

(i) *if* $p=\sigma(o)\notin W_G$ *is a smooth point of the hypersurface* $G$*, then*

$$\mathrm{codim}_o \, \mathrm{Bs}\,\Lambda_k \geq \mathrm{codim}_E \, \mathrm{Bs}\,\Lambda^E_k \geq \sharp[1,k]\cap\mathcal{M}+\sharp[1,k]\cap\mathcal{L},$$

*in particular,*

$$\dim_o \mathrm{Bs}\,\Lambda_\infty \leq 1,$$

(ii) *if* $p=\sigma(o)\notin W_G$ *is a double point of the hypersurface* $G$*, then*

$$\mathrm{codim}\,\mathrm{Bs}\,\Lambda_k \geq \mathrm{codim}_E \, \mathrm{Bs}\,\Lambda^E_k \geq \sharp[2,k]\cap\mathcal{M}+\sharp[2,k]\cap\mathcal{L},$$

*and moreover*

$$\mathrm{Bs}\,\Lambda_\infty \subset C_*$$

*(see Sec. 1.3). More to that, let* $P\subset\mathbb{P}$*,* $P\ni p$*, be an arbitrary hyperplane,* $P_F=\sigma^{-1}(P\cap G)$ *the corresponding section of the fiber* $F$*,* $\Lambda^P_k=\Lambda_k|_{P_F}$ *the restriction of the linear system* $\Lambda_k$ *onto* $P_F$*. Then for* $k\leq\max\{m,2l\}-2$

$$\mathrm{codim}_{P_F} \mathrm{Bs}\,\Lambda^P_k \geq \sharp[2,k]\cap\mathcal{M}+\sharp[2,k]\cap\mathcal{L},$$

*and*

$$\dim \mathrm{Bs}\,\Lambda^P_\infty \leq 1, \tag{13}$$

*whereas if in (13) the equality holds then the degree of the one-dimensional part of the basic subscheme* $\mathrm{Bs}\,\Lambda^P_\infty$ *does not exceed* $\lambda_{m,l}$*.*

(iii) *If* $p=\sigma(o)\in W_G$ *is a smooth point on the branch divisor* $W_G$*, then the following inequality holds:*

$$\mathrm{codim}_o \, \mathrm{Bs}\,\Lambda_k \geq \mathrm{codim}_E \, \mathrm{Bs}\,\Lambda^E_k \geq \sharp[1,k]\cap\mathcal{M}+1,$$

(iv) *if* $p=\sigma(o)\in W_G$ *is a double point on the branch divisor* $W_G$*, then the following inequality holds:*

$$\mathrm{codim}_o \, \mathrm{Bs}\,\Lambda_k \geq \mathrm{codim}_E \, \mathrm{Bs}\,\Lambda^E_k \geq \sharp[1,k]\cap\mathcal{M}.$$

**Proof.** To obtain out claims, we replace the hypertangent linear systems $\Lambda_k$ by their subsystems (9), (11) and (12), constructed above, and use the regularity conditions (Sec. 1.3,1.4). Q.E.D.

## 2.2 Scheme of the proof of the condition (vs)

Assume that there exists a prime divisor $Y\subset F=F_t$, satisfying the estimate

$$\frac{\mathrm{mult}_x \widetilde{Y}}{\deg Y} > \frac{1}{m}, \tag{14}$$

where $x\in E$ is an infinitely near point of the first order, that is, $E\subset\widetilde{F}$ is the exceptional divisor of the blow up of the point $o\in F$, $\varphi\colon\widetilde{F}\to F$. Here the singular

point $o \in F$ is generated by a singularity of the hypersurface $G = G_t$, that is, $p = \sigma(o) \in G$ is a non-degenerate double point, $p \notin W$. Set $\sigma^{-1}(p) = \{o, o^+\}$ and let $\varphi_G \colon \widetilde{G} \to G$ be the blow up of the point $p$. The map $\sigma$ extends in an obvious way to a morphism

$$\tilde{\sigma} \colon \widetilde{F} \setminus \{o^+\} \to \widetilde{G},$$

whereas on the exceptional divisor $E \subset \widetilde{F}$ the morphism $\tilde{\sigma}$ is an isomorphism, which makes it possible to identify $E$ with the exceptional divisor of the blow up $\varphi_G$ and thus consider $E$ as embedded in $\mathbb{T} = \mathbb{P}(T_p\mathbb{P}) \cong \mathbb{P}^M$, that is, in the exceptional divisor of the blow up

$$\varphi_{\mathbb{P}} \colon \widetilde{\mathbb{P}} \to \mathbb{P}$$

of the point $p \in \mathbb{P}$. Depending on the context one of the inclusions $E \subset \widetilde{F}$ or $E \subset \widetilde{G}$ will be meant.

Let us show that the assumption (14) leads to a contradictions. In order to do that, we will use the method developed in [20]. The arguments break into a few steps. The first step is given by

**Proposition 2.2.** *There exists a hyperplane $P \subset \mathbb{P}$, $P \ni p$, such that $\sigma(Y) \not\subset P$ and the effective algebraic cycle $Y_P = (Y \circ_F P_F)$, where $P_F = \sigma^{-1}(P_G)$, $P_G = P \cap G$ is a hyperplane section, satisfies the estimate*

$$\frac{\mathrm{mult}_o}{\deg} Y_P > \frac{3}{2m}.$$

The symbol $\circ_F$ is used to emphasize that the cycle $Y_P$ is constructed in the sense of the intersection theory on $F$, and not on $V$. For a proof of the proposition see [20].

Step two. Consider the variety $P_F \subset F$. It is an irreducible variety of dimension $M-1$ with the double point $o \in P_F$. Let $\varphi_P \colon \widetilde{P} \to P_F$ be the blow up of the point $o$, $E_P \subset \widetilde{P}$ the exceptional divisor. Obviously, $\widetilde{P}$ embeds into $\widetilde{F}$, and $E_P$ into $E$ as a hyperplane section of the quadric $E$ with respect to the embedding $E \hookrightarrow \mathbb{T}$. Since the variety $\widetilde{F}$ is factorial, the strict transform $\widetilde{Y}$ is a Cartier divisor. Therefore, the effective cycle $\widetilde{Y}_P = (\widetilde{Y} \circ \widetilde{P})$, that is, the strict transform of the cycle $Y_P$ on $\widetilde{P}$, is a Cartier divisor,

$$\widetilde{Y}_P \sim aH_P - bE_P,$$

where $H_P$ is the class of a hyperplane section. By Proposition 2.2

$$b > \frac{3}{2}a.$$

By the regularity condition we get for the tangent divisor

$$T = \sigma^{-1}(T_pG \cap G)$$

that $\mathrm{mult}_o T = 6$, $\deg T = 4m$, so that for the class of its strict transform $\widetilde{T} \subset \widetilde{F}$ we get $\widetilde{T} \sim 2H - 3E$ and thus for its restriction $\widetilde{T}_P = \widetilde{T} \cap \widetilde{P}$ on $\widetilde{P}$ we get

$$\widetilde{T}_P \sim 2H_P - 3E_P.$$

**Proposition 2.3.** *Let $Z \sim \alpha H_P - \beta E_P$ be an effective Cartier divisor on $\widetilde{P}$. Assume that $\beta > \frac{3}{2}\alpha$. Then $Z$ contains $\widetilde{T}_P$ as a component of positive multiplicity.*

**Proof** is given below.

Step three. Write down the effective divisor $\widetilde{Y}_P$ in the following form:

$$\widetilde{Y}_P = c\widetilde{T}_P + Z,$$

where $c \in \mathbb{Z}_+$ and the effective divisor $Z$ does not contain $\widetilde{T}_P$ as a component. Setting $Z \sim \alpha H_P - \beta E_P$, we obtain from the system of equations

$$a = 2c + \alpha, \quad b = 3c + \beta$$

and the condition $2b > 3a$, that

$$\beta > \frac{3}{2}\alpha.$$

By Proposition 2.3 this implies that $\widetilde{T}_P$ is a component of positive multiplicity of the divisor $Z$. A contradiction.

Thus we have proved that the estimate (14) is impossible which implies that the condition (vs) holds for the case of a singular point $o \in F$ outside the branch divisor. Q.E.D.

## 2.3 Movable families of curves

Let us prove Proposition 2.3. We use the method of the paper [20].

**Lemma 2.2.** *The divisor $T_P = T \cap P_F$ is swept out by a family of curves $\{C_\delta, \delta \in \Delta\}$, the general member of which is irreducible and satisfies the inequality*

$$\frac{\mathrm{mult}_o}{\deg} C_\delta > \frac{2}{3}. \tag{15}$$

First of all, let us obtain Proposition 2.3 from this fact. Let $\{\widetilde{C}_\delta, \delta \in \Delta\}$ be the strict transform of this family of curves on $\widetilde{P}$, $\widetilde{T}_P \subset \widetilde{P}$ the strict transform of the divisor $T_P$. Obviously,

$$(Z \cdot \widetilde{C}_\delta) = \alpha \deg C_\delta - \beta \,\mathrm{mult}_o\, C_\delta < 0,$$

since $\beta > \frac{3}{2}\alpha$. Therefore $\widetilde{C}_\delta \subset Z$. However the curves $\widetilde{C}_\delta$ sweep out $\widetilde{T}_P$, thus $Z \supset \widetilde{T}_P$. Q.E.D. for Proposition 2.3.

**Proof of Lemma 2.2.** The variety $P_F$ is of dimension $m + l - 2$, the divisor $T_P \subset P_F$ is of dimension $m + l - 3$. We construct the required family of curves $(C_\delta, \delta \in \Delta)$, intersecting $T_P$ with $m + l - 4$ hypertangent divisors. To order the construction procedure, let us introduce some new notations:

$$\mathcal{M} = \{2, \dots, m-1\}, \quad \mathcal{L} = \{l, \dots, 2l-1\},$$

$$c_e = \sharp[4, e] \cap \mathcal{M} + \sharp[3, e] \cap \mathcal{L}, \quad e \in \mathbb{Z}_+. \tag{16}$$

Here and below we assume silently that the segment $[a,b] \subset \mathbb{R}$ is the empty set when $b < a$. For $e \leq 2$ we get $c_e = 0$, for $e \geq \max\{m, 2l\} - 1$ we get that $c_e = m + l - 4$, provided that $l \geq 3$. Let us assume that this is the case and that $m \geq 4$. Note that for $m = 4$

$$\sharp[4, e] \cap \mathcal{M} = 0,$$

since this set is empty. The cases $l = 2$ and $m = 3$ we will treat separately. Obviously,

$$c_{e+1} \geq c_e.$$

Define the *ordering function*

$$\chi \colon \{1, \dots, m + l - 4\} \to \mathbb{Z}_+$$

by the formula

$$\chi([c_{e-1} + 1, c_e] \cap \mathbb{Z}_+) = e. \tag{17}$$

In accordance with our remark above, if $c_{e-1} = c_e$, then the formula (17) is meaningless, since the set $[c_{e-1} + 1, c_e]$ is empty. Note that

$$c_{e+1} - c_e \in \{0, 1, 2\}$$

by the definition (16). It is easy to check that (17) gives a correct definition of an integer-valued function $\chi$.

Denote by the symbol $\Lambda_i^P$ the restriction of the hypertangent system $\Lambda_i$ onto $P_F$. Set

$$\Lambda^P = \prod_{i=1}^{m+l-4} \Lambda_{\chi(i)}^P.$$

Note that in this product the hypertangent system $\Lambda_e$ can appear at most twice, see (16). Let

$$\mathbb{D} = \{D_i \in \Lambda_{\chi(i)}^P, \;\; i = 1, \dots, m + l - 4\} \in \Lambda^P$$

be a general set of hypertangent divisors.

**Definition 2.2.** We say that a family of closed algebraic sets $(\Gamma_u, u \in U)$ of (co)dimension $i$ on an algebraic variety $Z$ is a dense movable family if for a general $u \in U$ all irreducible components of the set $\Gamma_u$ are of (co)dimension $i$ and these components form a family of irreducible algebraic varieties sweeping out $Z$.

**Lemma 2.3.** *For $i = 1, \dots, m + l - 4$ the closed algebraic set*

$$R_i(\mathbb{D}) = \bigcap_{j=1}^{i} D_i \cap T_P$$

*is for a general $\mathbb{D} \in \Lambda^P$ of codimension $i$ in $T_P$. For $i = 1, \dots, m + l - 5$ the family of cycles*

$$(R_i(\mathbb{D}), \mathbb{D} \in \Lambda^P)$$

*is a dense movable family of cycles of codimension $i$ on $T$.*

**Proof.** Set $R_0(\mathbb{D}) = T$ and argue by induction on $i = 1, \ldots, m + l - 4$. Assume that the claim of the lemma is proved for $i \leq j \leq m + l - 5$ (if $j = 0$, then there is nothing to prove). Set $\chi(j + 1) = e$. By definition,

$$R_{j+1}(\mathbb{D}) = R_j(\mathbb{D}) \cap D_{j+1},$$

where $D_{j+1} \in \Lambda_e^P$ is a general divisor. By definition of the function $\chi$ we get

$$j + 1 \in [c_{e-1} + 1, c_e].$$

By Proposition 2.1, the following inequality holds:

$$\operatorname{codim}_{P_F} \operatorname{Bs} \Lambda_e^P \geq c_e + 1, \tag{18}$$

so that

$$\operatorname{codim}_{T_P} \operatorname{Bs} \Lambda_e^P|_{T_P} \geq c_e,$$

whereas

$$\operatorname{codim}_{T_P} R_j(\mathbb{D}) = j \leq c_e - 1$$

by (16), (17). Therefore, neither of the irreducible components of the closed subset $R_j(\mathbb{D})$ is contained in the base set of the hypertangent system $\Lambda_e^P$. In particular,

$$R_j(\mathbb{D}) \not\subset D_{j+1}$$

and therefore $R_{j+1}(\mathbb{D})$ is a closed subset of pure codimension $j + 1$ in $T_P$, which proves the first claim of the lemma. Now assume that $j \leq m + l - 6$. Then either

$$e \leq \max\{m, 2l\} - 2,$$

so that by Proposition 2.1 we get the estimate

$$\operatorname{codim}_{P_F} \operatorname{Bs} \Lambda_e^P \geq c_e + 2,$$

which is stronger than the inequality (18), or $e = \max\{m, 2l\} - 1$, but in this case $c_e = c_{e-1} + 2$, since

$$j + 2 \in [c_{e-1} + 1, c_e],$$

so that

$$\operatorname{codim}_{T_P} R_j(\mathbb{D}) = j = c_e - 2.$$

In any case for each irreducible component $Z$ of the set $R_j(\mathbb{D})$ for $j \leq m + l - 6$ we get

$$\operatorname{codim}_Z \operatorname{Bs}(\Lambda_e^P|_Z) \geq 2,$$

so that the linear system $\Lambda_e^P|_Z$ is movable. This proves the second claim of Lemma 2.3.

Consider the family of closed one-dimensional sets

$$(R(\mathbb{D}) = R_{m+l-4}(\mathbb{D}), \mathbb{D} \in \Lambda^P).$$

We can no longer assert that irreducible components of the set $R(\mathbb{D})$ form a movable family of curves: at the last step, that is, when we make curves from surfaces, some fixed components can appear. However, in any case the following decomposition holds:

$$R(\mathbb{D})=(T_P\circ D_1\circ\ldots\circ D_{m+l-4})=\sum_{\delta_i\in\Delta}C_{\delta_i}+\Phi, \tag{19}$$

where $(C_\delta,\delta\in\Delta)$ is a movable family of curves, $\Phi$ an effective 1-cycle, that is, the fixed part of the family of curves $R(\mathbb{D})$, $\mathbb{D}\in\Lambda^P$. We get the equality of 1-cycles

$$\Phi=\operatorname{Bs}\Lambda^P_\infty.$$

The family $(C_\delta,\delta\in\Delta)$ sweeps out $T_F$, *if it is non-empty*. However, by construction

$$\deg R(\mathbb{D})=4m\prod_{j=1}^{m+l-4}\chi(j)=4m\left(\prod_{j=4}^{m-1}j\right)\left(\prod_{j=l}^{2l-1}j\right)=$$

$$=\frac{2m!(2l-1)!}{3(l-1)!},$$

whereas by the regularity condition

$$\deg\Phi<\lambda_{m,l}=\frac{m!(2l-1)!}{6(l-1)!}<\deg R(\mathbb{D}).$$

Therefore, the family of irreducible curves $(C_\delta,\delta\in\Delta)$ is non-empty and sweeps out the divisor $T$.

Let us, finally, estimate the ratio $\operatorname{mult}_o/\deg$ for a general curve $C_\delta$. As we mentioned above, $\operatorname{mult}_o\Phi=\deg\Phi$ (see Sec. 1.3). Besides, for a general set $\mathbb{D}\in\Lambda^P$ the ratio

$$\frac{\operatorname{mult}_o}{\deg}C_{\delta_i}$$

(in the sense of the formula (19)) does not depend on $i$. Consequently,

$$\frac{\operatorname{mult}_o}{\deg}C_\delta=\frac{\operatorname{mult}_o R(\mathbb{D})-\deg\Phi}{\deg R(\mathbb{D})-\deg\Phi}.$$

However, by construction

$$\operatorname{mult}_o R(\mathbb{D})\geq 6\prod_{j=1}^{m+l-4}(\chi(j)+1)=6\left(\prod_{j=5}^{m}j\right)\left(\prod_{j=l+1}^{2l}j\right)=$$

$$=\frac{m!}{4}\cdot\frac{(2l)!}{l!},$$

whence

$$\frac{\operatorname{mult}_o}{\deg}C_\delta\geq\frac{\dfrac{m!}{4}\cdot\dfrac{(2l)!}{l!}-\lambda_{m,l}}{\dfrac{2m!}{3}\cdot\dfrac{(2l-1)!}{(l-1)!}-\lambda_{m,l}}>\frac{2}{3}$$

in accordance with the choice of the number $\lambda_{m,l}$. This proves the lemma for $m \geq 5$, $l \geq 3$.

If $l = 2$, then the arguments presented above work with the only modification: instead of (16) one should use the formula

$$c_e = \sharp[3, e] \cap \mathcal{M} + \sharp[3, e] \cap \mathcal{L},$$

$e \in \mathbb{Z}_+$. In this case an independent hypertangent divisor adds into the linear system $\Lambda^P_2$ and thus the codimension of its base set (and the codimension of the base set of all the subsequent hypertangent systems $\Lambda^P_j$, $j \geq 3$) exceeds by one the corresponding codimension in the just considered case $l \geq 3$. It is this fact that makes it possible to change the definition of the number $c_e$ and accordingly shift by one the function $\chi$. The rest of the arguments are completely similar to the case $l \geq 3$ discussed above.

The case $m \geq 3$ is slightly harder. In order to obtain the needed codimension of the base set of a hypertangent system, one should use the following set of hypertangent divisors:

$$\mathbb{D} = \{D_i \in \Lambda^P_{l+i} \mid i = 1, \dots, l-1\}.$$

We draw the reader's attention to the fact that the first divisor in this set is taken from the linear system $\Lambda^P_{l+1}$, that is, in contrast to the case $m \geq 4$, which we considered above, we skip the system $\Lambda^P_l$. As a result, we obtain once again a movable family of closed algebraic sets

$$R_k(\mathbb{D}) = \left(\bigcap_{j=1}^{k} D_j\right) \cap T$$

for $k \leq 2l-2$, whereas irreducible components of the sets $R_k(\mathbb{D})$ form a family and sweep out $T$. Again we modify the family of curves $R_{2l-1}(\mathbb{D})$, deleting the fixed part $\Phi$ of degree $\deg\Phi < \lambda_{3,l} = 12(l-2)\dfrac{(2l-1)!}{(l+1)!}$ and obtain a family of irreducible curves $(C_\delta, \delta \in \Delta)$, sweeping out $T$ and satisfying the estimate (15). Proof of Lemma 2.2 is now complete.

# 3 Singularity of a fiber on the branch divisor

## 3.1 Notations and discussion of the regularity condition

We have the double cover

$$F = F_t \xrightarrow{\sigma} G = G_t \subset \mathbb{P} = \mathbb{P}^{M+1},$$

$G \subset \mathbb{P}$ is a smooth hypersurface of degree $m \leq M-1$. At the point $p \in G$ the branch hypersurface $W_G = W \cap G$ has an isolated quadratic singularity, so that $o = \sigma^{-1}(p) \in F$ is an (isolated) non-degenerate double point of the fiber $F$. We get

the following commutative diagram of maps

$$
\begin{array}{ccccccc}
E = E_F & \subset & \widetilde{F} & \stackrel{\tilde\sigma}{\to} & \widetilde{G} & \supset & E_G \\
& \varphi_F & \downarrow & & \downarrow & \varphi_G & \\
& & F & \stackrel{\sigma}{\to} & G, & &
\end{array}
$$

where $\varphi_F$ and $\varphi_G$ are the blow ups of the points $o \in F$ and $p \in G$, respectively, $E_F$ and $E_G$ are the exceptional divisors, $\tilde\sigma$ the double cover, branched over $\widetilde{W}_G \subset \widetilde{G}$, that is, over the strict transform of the divisor $W_G$. Besides,

$$\tilde\sigma_E = \tilde\sigma|_E \colon E \to E_G \cong \mathbb{P}^{M-1}$$

is the double cover, branched over the quadric

$$W_E = \widetilde{W}_G \cap E_G.$$

The symbol $H_E$ stands for the hyperplane section of the quadric $E$ with respect to the standard embedding $E \hookrightarrow \mathbb{P}^M$, $\operatorname{Pic} E = \mathbb{Z} H_E$.

Let $W_t = W \cap \mathbb{P}_t$ be given by the equation

$$h = w_1 + w_2 + \ldots + w_{2l} = 0,$$

and $G$ by the equation

$$f = q_1 + q_2 + \ldots + q_m = 0$$

with respect to the affine coordinates $z_* = (z_1, \ldots, z_{M+1})$ with the origin at the point $p$. The divisor $W_G$ has at $p$ a non-degenerate quadratic singularity, so that $w_1 = \lambda q_1$, for simplicity of notations assume that $q_1 = z_{M+1}$. The quadratic polynomial $\bar w_2 = w_2|_{\{z_{M+1}=0\}}$ is of the maximal rank. Take $z_1, \ldots, z_M$ for homogeneous coordinates on $E_G$, then

$$\tilde\sigma_E \colon E \to E_G \cong \mathbb{P}^{M-1}$$

is branched over the non-singular quadric $W_E = \{\bar w_2 = 0\}$. For an arbitrary point $y \in E_G \setminus W_E$ let $C(y) \subset E_G$ be the cone consisting of all lines $L \subset E_G$ that contain $y$ and touch $W_E$. More formally, let

$$\pi_y \colon E_G \setminus \{y\} \to \mathbb{P}^{M-2}$$

be the projection from the point $y$. Its restriction onto the quadric $W_E$,

$$\pi_y|_{W_E} \colon W_E \to \mathbb{P}^{M-2}$$

is a double cover, branched over a quadric $Q(y) \subset \mathbb{P}^{M-2}$. Now

$$C(y) = \overline{\pi_y^{-1}(Q(y))}.$$

Obviously, $C(y)$ is a quadric cone with the vertex at the point $y$. Since the quadric $W_E$ is non-singular, the cone $C(y)$ has only one singularity, that is, the point $y$.

Denote the restriction of the polynomial $q_i$ onto the hyperplane $q_{M+1} = 0$ by the symbol $\bar{q}_i$. By the regularity condition, the system of homogeneous equations

$$\bar{q}_2 = \ldots = \bar{q}_m = 0$$

defines in $E_G$ an irreducible reduced complete intersection of codimension $(m-1)$, an irreducible subvariety $Z_{2\cdot\ldots\cdot m}$. Moreover, the quadric $\bar{q}_2 = 0$ is smooth and distinct from $W_E$.

**Lemma 3.1.** *Assume that the condition* (R2.2) *holds. Then the subvariety $Z_{2\cdot\ldots\cdot m}$ is not contained in a quadric cone $C(y)$, $y \in E_G \backslash W_E$, and in a tangent plane $T_y W_E$, $y \in W_E$.*

**Proof.** Set

$$Z_{2\cdot\ldots\cdot j} = \{z \in \mathbb{P}^{M-1} \mid \bar{q}_2 = \ldots = \bar{q}_j = 0\}.$$

It is easy to see that $Z_{2\cdot\ldots\cdot j}$ is an irreducible reduced complete intersection of codimension $j$. From the long exact cohomology sequence we obtain that

$$h^0(\mathcal{O}_{Z_2}(2)) = \ldots = h^0(\mathcal{O}_{Z_{2\cdot\ldots\cdot j}}(2)) = \ldots = h^0(\mathcal{O}_{Z_{2\cdot\ldots\cdot m}}(2)),$$

and moreover, the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^{M-1}}(2)) \to H^0(\mathcal{O}_{Z_{2\cdot\ldots\cdot m}}(2))$$

is surjective. This implies that $Z_{2\cdot\ldots\cdot m}$ is contained in one and only one quadric $Z_2$ and thus is not contained in any quadric cone $C(y)$, $y \in E_G \setminus W_E$. In a similar way, the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^{M-1}}(1)) \to H^0(\mathcal{O}_{Z_{2\cdot\ldots\cdot m}}(1))$$

is an isomorphism, so that $Z_{2\cdot\ldots\cdot m}$ is not contained in a hyperplane, in particular, in a hyperplane of the form $T_y W_E$, $y \in W_E$.

Now fix a prime divisor $R \subset F$, and let $\widetilde{R} \subset \widetilde{F}$ be its strict transform. Fix also an arbitrary point $x \in E$, lying outside the branch divisor of the cover $\tilde{\sigma}_E$, that is, $\tilde{\sigma}(x) \notin W_E$. (For a point $x \in E$ on the branch divisor the arguments given below work automatically with simplifications. The arguments of Sec. 2 can be also used in this case, in contrast to the situation outside the branch divisor $W_E$.)

**Proposition 3.1.** *The following estimate holds:*

$$\mu = \operatorname{mult}_x \widetilde{R} \leq \frac{1}{m} \deg R. \tag{20}$$

**Remark.** For some $k \geq 1$ we have $R \sim H_F$, where $H_F = \sigma^* H_G$ is a hyperplane section. Since obviously $\deg R = 2mk$, the estimate (20) takes the form of the following inequality:

$$\mu \leq 2k.$$

## 3.2 Start of the proof of the condition (vs)

Assume the converse: $\mu > 2k$. We have the presentation

$$\widetilde{R} \sim k\varphi_F^* H_F - \nu E,$$

whereas $\mathrm{mult}_o R = 2\nu$.

**Lemma 3.2.** *The following inequality holds: $\nu \leq 2k$.*

**Proof.** Assume the converse: $\nu > 2k$. Then

$$\frac{\mathrm{mult}_o}{\deg} R > \frac{2}{m}.$$

Set $\bar{R} = \sigma(R) \subset G$. It is a prime divisor on the smooth hypersurface $G \subset \mathbb{P}$. Since $\sigma\colon R \to \bar{R}$ is a finite morphism, we get the inequality

$$\frac{\mathrm{mult}_o}{\deg} \bar{R} > \frac{2}{m}.$$

However, this is impossible, since $p \in G$ is a regular point. Indeed, the tangent divisor $T_1^+ = T_p G \cap G$ satisfies the equality

$$\frac{\mathrm{mult}_p}{\deg} T_1^+ = \frac{2}{m},$$

so that $\bar{R} \neq T_1^+$ and $(\bar{R} \circ T_1^+)$ is an effective cycle of codimension two ($T_1^+$ is obviously irreducible).

Since $\mathrm{mult}_p T_1^+ = 2$, we get the inequality

$$\mathrm{mult}_p(\bar{R} \circ T_1^+) \geq 2 \, \mathrm{mult}_p \bar{R}.$$

Taking into account that $\deg(\bar{R} \circ T_1^+) = \deg \bar{R}$, we conclude that there exists an irreducible component $Y_2$ of the cycle $(\bar{R} \circ T_1^+)$, satisfying the estimate

$$\frac{\mathrm{mult}_p}{\deg} Y_2 \geq 2 \frac{\mathrm{mult}_p}{\deg} \bar{R}.$$

As usual, let $f = q_1 + q_2 + \ldots + q_m$ be the polynomial defining the hypersurface $G$ with respect to the coordinate system $z_*$ with the origin at the point $p$. Setting

$$f = q_1 + q_2 + \ldots + q_i,$$

let us construct the hypertangent systems

$$\Lambda_i^G = \left| \sum_{j=1}^{i} f_j s_{i-j}|_G = 0 \right|,$$

and consider the standard hypertangent divisors

$$T_i^+ = \{ f_i|_G = 0 \} \in \Lambda_i^G.$$

Set

$$T_i = \sigma^* T_i^+, \quad \Lambda_i = \sigma^* \Lambda_i^G.$$

These divisors and linear systems will be of crucial importance below. At the moment, note that by the regularity condition we get

$$\operatorname{codim}_G \operatorname{Bs} \Lambda_i^G = i$$

(in fact, $\operatorname{Bs} \Lambda_i^G = T_1^+ \cap \ldots \cap T_i^+$). Let

$$\mathbb{D} = (D_1, \ldots, D_{m-1}) \in \prod_{j=1}^{m-1} \Lambda_i^G$$

be a general set of divisors. Let us construct by induction a sequence of irreducible subvarieties $Y_i$, $i = 1, \ldots, m-1$, satisfying the following properties:

(i) $Y_1 = \bar{R}$, $Y_2$ was constructed above, $\operatorname{codim}_G Y_i = i$;

(ii) $Y_{i+1} \subset Y_i$, $Y_i \not\subset D_{i+1}$, $Y_{i+1}$ is an irreducible component of the closed set $Y_i \cap D_{i+1}$;

(iii) the estimate

$$\frac{\operatorname{mult}_p}{\deg} Y_{i+1} \geq \frac{i+2}{i+1} \cdot \frac{\operatorname{mult}_p}{\deg} Y_i$$

holds.

It is possible to construct this sequence because

$$\operatorname{codim}_G \Lambda_{i+1}^G = i+1 > \operatorname{codim}_G Y_i,$$

so that for a general divisor $D_{i+1} \subset \Lambda_{i+1}^G$ we have $Y_i \not\subset D_{i+1}$. One can ensure that the property (iii) holds since $\Lambda_j^G \subset |jH_G|$ and $\operatorname{mult}_p \Lambda_j^G = j+1$.

Now for an irreducible subvariety $Y = Y_{m-1}$ we get the estimate

$$1 \geq \frac{\operatorname{mult}_p}{\deg} Y \geq \underbrace{\frac{m}{m-1} \cdot \frac{m-1}{m-2} \cdot \ldots \cdot \frac{4}{3} \cdot \frac{2}{1}}_{\| \atop \frac{2m}{3}} \cdot \frac{\operatorname{mult}_p}{\deg} \bar{R},$$

whence we get

$$\frac{\operatorname{mult}_p}{\deg} \bar{R} \leq \frac{3}{2m}.$$

Therefore the ratio $\operatorname{mult}_p / \deg$ attains its maximum at the tangent divisor $T_p G \cap G$ and this maximum is equal to 2. A contradiction. Q.E.D. for the lemma.

## 3.3 Hypertangent divisors and tangent cones

Thus $\nu \leq 2k < \mu$. On the other side,

$$\mu = \mathrm{mult}_x \widetilde{R} \leq \mathrm{mult}_x(\widetilde{R} \circ E) \leq \deg(\widetilde{R} \circ E) = 2\nu.$$

Set $B = T_x E \cap E$, where the quadric $E$ is considered as embedded in $\mathbb{P}^M$ in the standard way. By Lemma 5 from Sec. 3.5 in [20],

$$\mathrm{mult}_B \widetilde{R} \geq \frac{1}{2}(\mu - \nu),$$

whereas for the effective cycle $R_E = (\widetilde{R} \circ E)$ we get

$$\mathrm{mult}_B R_E \geq \mu - \nu. \tag{21}$$

Set $\widetilde{T}_i, \widetilde{\Lambda}_i$ to be the strict transforms of the divisors $T_i$ and linear systems $\Lambda_i$ on the blow up $\widetilde{F}$ of the fiber $F$. It is easy to see that $\widetilde{T}_i \subset \widetilde{\Lambda}_i$, since by construction

$$\mathrm{mult}_o \Lambda_i = \mathrm{mult}_o T_i.$$

Set also

$$\mathbb{T}_i = (\widetilde{T}_i \circ E) = \widetilde{T}_i \cap E$$

to be the projectivized tangent cone to the divisor $T_i$ at the point $o$. Recall that the quadric $E$ is realized as the double cover $\tilde{\sigma}_E \colon E \to E_G \cong \mathbb{P}^{M-1}$, branched over the quadric $W_E$. For a system $(z_1, \ldots, z_{M+1})$ of affine coordinates on $\mathbb{P}$ with the origin at the point $p$ we may assume that $q_1 \equiv z_{M+1}$ and therefore $(z_1, \ldots, z_M)$ can be taken for homogeneous coordinates on the projective space $E_G$. In terms of these coordinates the hypersurface $\mathbb{T}_i \subset E$ is given by the equation

$$(\tilde{\sigma}_E)^* q_{i+1}|_{E_G}.$$

Finally, set

$$\Lambda_i^E = \widetilde{\Lambda}_i|_E$$

to be the projectivized tangent system of the linear system $\Lambda_i$ at the point $o$. Equations of divisors of this linear system are obtained by pulling back to $E$ via $\tilde{\sigma}_E$ the equations

$$\sum_{j=1}^{i} \bar{q}_{j+1} \bar{s}_{i-j}, \tag{22}$$

where $\bar{\sharp}$ means the restriction of the polynomial $\sharp$ onto the hyperplane $z_{M+1} = 0$. Obviously,

$$\mathbb{T}_i \sim (i+1)H_E, \quad \Lambda_i^E \subset |(i+1)H_E|,$$

besides the equations (22) imply directly that

$$\mathrm{Bs}\,\Lambda_i = T_1 \cap \ldots \cap T_i, \quad \mathrm{Bs}\,\Lambda_i^E = \mathbb{T}_1 \cap \ldots \cap \mathbb{T}_i,$$

both equalities in the scheme-theoretic sense.

## 3.4 Constructing new cycles

By the regularity condition the set $\mathbb{T}_1 \cap \ldots \cap \mathbb{T}_i$ is irreducible and not contained in the divisor $B$ for all $i = 1, \ldots, m-1$. Let

$$\mathcal{L} = (L_2, \ldots, L_{m-1}) \in \Lambda_2 \times \ldots \times \Lambda_{m-1}$$

be a general set of hypertangent divisors. We denote the strict transform of the hypertangent divisor $L_j$ on $\widetilde{F}$ by the symbol $\widetilde{L}_j$ and its projectivized tangent cone at the point $o \in F$ by the symbol

$$\mathbb{L}_j = (\widetilde{L}_j \circ E).$$

For a general divisor $L_j \in \Lambda_j$ we get $\mathbb{L}_j = \widetilde{L}_j \cap E$.

**Lemma 3.3.** (i) *Let $Y \subset F$ be a fixed irreducible subvariety of codimension $l \leq m-2$. For a general divisor $L_{l+1} \in \Lambda_{l+1}$ we have $Y \not\subset L_{l+1}$.*

(ii) *Let $Y \subset E$ be a fixed irreducible subvariety of codimension $l \leq m-2$. For a general divisor $L_{l+1} \in \Lambda_{l+1}$ we have $Y \not\subset \mathbb{L}_{l+1}$.*

**Proof.** By the regularity condition

$$\operatorname{codim}_F \operatorname{Bs} \Lambda_{l+1} = l+1, \quad \operatorname{codim}_E \operatorname{Bs} \Lambda^E_{l+1} = l+1$$

and for a general divisor $L_j \in \Lambda_j$ we have $\mathbb{L}_j \in \Lambda^E_j$. Q.E.D. for the lemma.

**Corollary 3.1.** *For a general set $\mathcal{L}$ we have*

$$\operatorname{codim}_F(R \cap L_2 \cap \ldots \cap L_{m-1}) = m-1,$$

$$\operatorname{codim}_E(R_E \cap \mathbb{L}_2 \cap \ldots \cap \mathbb{L}_{m-1}) = m-1.$$

From this fact we obtain that the following effective algebraic cycles of codimension $m-1$ are well defined on $F$ and $E$, respectively:

$$R^+ = (R \circ L_2 \circ \ldots \circ L_{m-1})$$

and

$$R^+_E = (R_E \circ \mathbb{L}_2 \circ \ldots \circ \mathbb{L}_{m-1}),$$

whereas (for a general set $\mathcal{L}$)

$$R^+_E = (\widetilde{R}^+ \circ E)$$

is the projectivized tangent cone to the cycle $R^+$ at the point $o$. Let us describe the structure of these effective cycles. First of all we get

$$\deg R^+ = 2km \cdot (m-1)! = 2km!,$$

$$\operatorname{mult}_o R^+ = \deg R^+_E = 2\nu \cdot 3 \cdot \ldots \cdot m = \nu m!.$$

**Lemma 3.4.** *Let $Y$ be an irreducible component of the cycle $R^+$. If $Y \subset T_1$, then*

$$Y = T_1 \cap T_2 \cap \ldots \cap T_{m-1}.$$

**Proof.** By construction, the equation of the divisor $L_i$ is of the form

$$f_1 s_{i-1} + f_2 s_{i-2} + \ldots + f_i s_0,$$

where $s_j$ is a homogeneous polynomial of degree $j$ in the coordinates $z_*$. Since the hypertangent divisors $L_i$ are assumed to be general, we may assume that $s_0 \neq 0$ and thus normalize the equation by the condition that $s_0 = 1$. Assume that $Y \subset T_1$. Then the following polynomials vanish on $Y$:

$$\begin{array}{llllllll}
f_1, & & & & & & & \\
f_1 s_{2,1} & + & f_2, & & & & & \\
f_1 s_{3,2} & + & f_2 s_{3,1} & + & f_3, & & & \\
 & & & & \ldots & & & \\
f_1 s_{m-1,m-2} & + & & \ldots & & + \; f_{m-2} s_{m-1,1} & + & f_{m-1},
\end{array}$$

where $s_{i,j}$ is a homogeneous polynomial of degree $j$. Thus

$$f_1|_Y \equiv f_2|_Y \equiv \ldots \equiv f_{m-1}|_Y \equiv 0,$$

so that $Y \subset T_1 \cap T_2 \cap \ldots \cap T_{m-1}$, but the latter set is irreducible and of the same dimension as $Y$. This proves Lemma 3.4.

## 3.5 Degrees and multiplicities

Set

$$T = T_1 \cap T_2 \cap \ldots \cap T_{m-1}, \quad \mathbb{T} = \mathbb{T}_1 \cap \mathbb{T}_2 \cap \ldots \cap \mathbb{T}_{m-1}.$$

Taking into consideration that $T = (T_1 \circ \ldots \circ T_{m-1})$ and $\mathbb{T} = (\mathbb{T}_1 \circ \ldots \circ \mathbb{T}_{m-1})_E$, it is easy to verify that

$$\deg T = \operatorname{mult}_o T = \deg \mathbb{T} = 2m!.$$

Now write down

$$R^+ = aT + R^\sharp, \quad R^+_E = a\mathbb{T} + R^\sharp_E, \tag{23}$$

where $a \in \mathbb{Z}_+$, the effective cycle $R^\sharp$ is uniquely defined by the condition that it does not contain the subvariety $T$ as a component, and

$$R^\sharp_E = (\widetilde{R}^\sharp \circ E)$$

is the projectivized tangent cone to $R^\sharp$ at the point $o$. Note that the irreducible subvariety $\mathbb{T}$, generally speaking, can come into the effective cycle $R^\sharp_E$ as a component.

**Lemma 3.5.** *The following estimate holds:*

$$2 \operatorname{mult}_o R^\sharp \leq \deg R^\sharp.$$

**Proof.** Let $Y$ be an irreducible component of the cycle $R^\sharp$. By construction, $Y \neq T$; therefore by Lemma 3.4 $Y \not\subset T_1$. Thus the closed subset

$$T_1 \cap \operatorname{Supp} R^\sharp$$

is of codimension $m$, so that the effective cycle

$$R^* = (R^\sharp \circ T_1)$$

is well defined. Now we have a standard chain of estimates:

$$2\,\mathrm{mult}_o R^\sharp \leq \mathrm{mult}_o R^* \leq \deg R^* = \deg R^\sharp,$$

which is what we need.

As in Corollary 3.1, Lemma 3.5 implies that the set

$$B \cap \mathbb{L}_2 \cap \ldots \cap \mathbb{L}_{m-1}$$

is of codimension $m-1$ in $E$. Denote by $B^+$ the part of the effective equidimensional cycle $R_E^+$, the support of which is contained in $B$:

$$R_E^+ = \sum_{i\in I} r_i Y_i, \quad B^+ = \sum_{i\in I, Y_i\subset B} r_i Y_i.$$

**Lemma 3.6.** *The following estimate holds:*

$$\deg B^+ \geq (\mu - \nu) m!$$

**Proof.** Indeed, by (21) we get

$$R_E = (\mu - \nu) B + \Delta,$$

where $\Delta$ is an effective cycle. Furthermore,

$$\deg(B \circ \mathbb{L}_2 \circ \ldots \circ \mathbb{L}_{m-1}) = 2 \cdot 3 \cdot \ldots \cdot m = m!,$$

which proves the lemma.

**Lemma 3.7.** *Let $Y \subset \mathbb{L}_2 \cap \ldots \cap \mathbb{L}_{m-1}$ be an irreducible subvariety of codimension $m-1$ in $E$. If $Y \subset \mathbb{T}_1$, then $Y = \mathbb{T}$.*

**Proof.** The equation of the divisor $\mathbb{L}_i$ with respect to the homogeneous coordinates $z_*$ is of the form

$$q_2 s_{i-1} + \ldots + q_{i+1},$$

where $s_j$ is a homogeneous polynomial of degree $j$. If $Y \subset \mathbb{T}_1$, then the following polynomials vanish on $Y$:

$$\begin{array}{lllllll} q_2, & & & & & & \\ q_2 s_{2,1} & + & q_3, & & & & \\ & & & \ldots & & & \\ q_2 s_{m-1,m-2} & + & \ldots & + & q_{m-1} s_{m-1,1} & + & q_m, \end{array}$$

where $\deg s_{i,j} = j$. Consequently,

$$q_2|_Y \equiv q_3|_Y \equiv \ldots \equiv q_m|_Y \equiv 0,$$

that is, $Y \subset \mathbb{T}$, and since the dimensions coincide, $Y = \mathbb{T}$. Q.E.D. for the lemma.

**Corollary 3.2.** *None of the components of the closed set*

$$B \cap \mathbb{L}_2 \cap \ldots \cap \mathbb{L}_{m-1}$$

*is contained in* $\mathbb{T}_1$.

**Proof.** Let $Y$ be such component and $Y \subset \mathbb{T}_1$. By the previous lemma, $Y = \mathbb{T}$. Thus $\mathbb{T} \subset B$: a contradiction with the regularity condition. Q.E.D. for the corollary.

Let us complete, at long last, the proof of Proposition 3.1. From the presentations (23) we get

$$\deg R^+ = 2km! = 2am! + \deg R^\sharp,$$

$$\operatorname{mult}_o R^+ = \nu m! = 2am! + \operatorname{mult}_o R^\sharp.$$

By Corollary 3.2 the effective cycle $B^+$ lies entirely in $R^\sharp_E$. In particular,

$$\deg R^\sharp_E \geq \deg B^+ \geq (\mu - \nu)m!. \tag{24}$$

However, $\deg R^\sharp_E = \operatorname{mult}_o R^\sharp$. Applying Lemma 3.5, we obtain:

$$2(\nu m! - 2am!) \leq 2km! - 2am!.$$

Let us rewrite the inequality (24) in the form

$$\nu m! - 2am! \geq (\mu - \nu)m!.$$

Easy computations give us the two inequalities

$$k + a \geq \nu,$$

$$2\nu - 2a \geq \mu,$$

which imply the desired estimate (20) in an obvious way.

However, we assumed that $\mu > 2k$. The contradiction completes our proof of Proposition 3.1 and Theorem 1 as well.

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